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AN INTRINSIC CONSTRUCTION OF STEENROD SQUARES FOR MOD 2 DE RHAM FORMS

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Let A^{**} be the simplicial bigraded (by weight and dimension) algebra of differential forms with coefficients in $\mathbb{Z}/2$ as described in [1–3], or [6]. In [1] and [2] we computed the cohomology of the cochain operations associated to A^{**}, in terms of the mod 2 Steenrod algebra, $\mathscr{A}$. For example, the algebra of 'stable' cohomology operations associated to this theory is a Laurent series $\mathscr{A}[\sigma,\sigma^{-1}]$ where σ is the canonical cohomology operation raising weight by one, see [1] or [2]. The techniques are those of Kristensen [5] and use the de Rham theorems of Cartan [3] or Miller [6]; briefly stated, a cochain operation which commutes with the differential on A^{**} gives rise to a 'stable' cohomology operation and a 'stable' cohomology operation may be lifted to give such a cochain operation; one can then describe the stable cohomology operations via the de Rham theorem. It is reasonable to ask for an intrinsic description of the operations on this theory, that is, a description which makes no appeal to the de Rham theorem. In this paper, we give such a description.

Recall the construction of A^{**} as described in [1], [3], or [6]. Let R be any commutative ring with unit and let $\Gamma(x)=\bigoplus_{r\geq 0}\Gamma_r(x)$ be the divided power algebra over R on a single generator x, that is, $\Gamma_r(x)$ is the free R-module on a single generator $\gamma_r(x)$; multiplication is defined by

$$\gamma_r(x)\cdot\gamma_s(x)=\frac{(r+s)!}{r!\,s!}\,\gamma_{r+s}(x).$$

Now let $\Gamma(x;\mathrm{d}x)$ denote the Koszul complex $\Gamma(x)\otimes E(\mathrm{d}x)$ where $E(\mathrm{d}x)$ denotes the exterior algebra over R on a single generator $\mathrm{d}x$. Then $\Gamma(x;\mathrm{d}x)$ is a commutative differential algebra with $\mathrm{d}(\gamma_r(x))=\gamma_{r-1}(x)\,\mathrm{d}x$ and $\mathrm{d}(\mathrm{d}x)=0$, and is bigraded by declaring $\gamma_r(x)$ to be of dimension zero, weight r and $\gamma_r(x)\,\mathrm{d}x$ to be of dimension one and weight $r+1$. Let $\Gamma(x_0,\dots,x_n;\mathrm{d}x_0,\dots,\mathrm{d}x_n)$ denote the tensor product of the $\Gamma(x_i;\mathrm{d}x_i)$; this is a bigraded commutative differential algebra, and A_n^{**} is

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$\Gamma(x_0, \ldots, x_n; dx_0, \ldots, dx_n)$ modulo the ideal generated by $\sum_{i=0}^n dx_i$. The collection of all these A_n^{**} for $n \geq 0$ is denoted A^{**} and called the de Rham algebra over R; it is a simplicial bigraded commutative differential algebra over R. The simplicial structure is obtained by thinking of A_n^{**} as living on the standard n-simplex and using the induced face and degeneracy operators. In this paper we take $R = \mathbb{Z}/2$. If X is any simplicial set, we define $A^{n,*}X = M(X, A^{n,*})$ = the set of simplicial maps from X to the underlying simplicial set of $A^{n,*}$ and use the induced differential and multiplication to obtain the mod 2 de Rham cohomology of X.

Let $Z^{n,*}A$ denote the simplicial vector space obtained as the kernel of the differential $d: A^{n,*} \to A^{n+1,*}$. Then we have the principal Kan fibrations

$$Z^{n,*}A \rightarrowtail A^{n,*} \twoheadrightarrow Z^{n+1,*}A.$$

These fibrations are of no use in a construction of cochain operations giving rise to the Steenrod squares because, for example, since $A^{n,m} = 0$, for $n > m$ we have no Künneth or Eilenberg–Zilber theorem for this theory.

Now consider the following principal Kan fibrations

$$Z^{n,n}A \xrightarrow{i^n} CZ^{n,n}A \xrightarrow{p^n} SZ^{n,n}A,$$

where $CZ^{n,n}A$ denotes the (contractible) cone on $Z^{n,n}A$ and $SZ^{n,n}A$ denotes the suspension of $Z^{n,n}A$ (see Curtis [4, p. 137]).

Lemma. *There is a linear isomorphism* $r: SZ^{n,n}A \cong Z^{n+1,n+1}A$ *for all* $n \geq 0$.

Proof. There are a number of ways to see this. First, it is possible to write down explicit linear isomorphisms for the two simplicial vector spaces in question (the isomorphism $SZ^{0,0}A = Z^{1,1}A$ has a slightly different form from the rest). On the other hand, it is not hard to see that each $Z^{n,n}A$ is a minimal complex in the sense of May [7], and they are also Eilenberg–MacLane complexes (of type $(\mathbb{Z}/2, n)$ see Cartan [1], Miller [6] or Campbell [2]; we are forgetting the divided power structure on $\pi_n(Z^{n,*}A)$). Now suspension preserves minimality (easy to see); it is not hard to see that $SZ^{n,n}A$ is an Eilenberg–MacLane complex of the same type as $Z^{n+1,n+1}A$ and since they are both minimal, they are isomorphic, see May [7, p. 100]. □

Remark. The usual construction of an Eilenberg–MacLane complex in the category of simplicial sets is also minimal, usually denoted $K(\mathbb{Z}/2, n)$, and there is a principal Kan fibration

$$K(\mathbb{Z}/2, n) \rightarrowtail L(\mathbb{Z}/2, n+1) \twoheadrightarrow K(\mathbb{Z}/2, n+1),$$

in which the total space is contractible, see May [7]. Now according to Cartan [1] or Miller [6] there is a map called 'integration' which we denote $I: Z^{n,*}A \to K(\mathbb{Z}/2, n)$ inducing isomorphisms of homotopy groups, consequently $I: Z^{n,n}A \to K(\mathbb{Z}/2, n)$ is a linear isomorphism since both simplicial vector spaces are minimal. It is not hard to build an extension of the integration map to a linear map still denoted $I: CZ^{n,n}A \to L(\mathbb{Z}/2, n+1)$ such that the diagram below commutes

$$\begin{array}{ccccc}
Z^{n,n}A & \rightarrowtail & CZ^{n,n}A & \xrightarrow{rp^n} & Z^{n+1,n+1}A \\
\downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} \\
K(\mathbb{Z}/2, n) & \rightarrowtail & L(\mathbb{Z}/2, n+1) & \twoheadrightarrow & K(\mathbb{Z}/2, n+1)
\end{array}$$

Since the outside two maps are isomorphisms, so is the middle map, by the five lemma. It follows that all the constructions (cup-product, Steenrod squares, Eilenberg–Zilber theorem, and Künneth theorems) that are available on the usual singular cochains may be pased to the forms via this diagram. In fact, one can also reprove these theorems for the cochain complex

$$CZ^{0,0}A \to \cdots \to CZ^{n,n}A \to CZ^{n+1,n+1}A \to \cdots,$$

just by mimicking the usual proofs.

By way of example, we show how to construct cochain operations

$$\mathrm{sq}^i : CZ^{n,n}A \to CZ^{m,m}A, \qquad n \geq 0, \quad m = n + i$$

which pass to the Steenrod squares in cohomology. Given a simplicial set X of finite type we form the cochain complex E^* where $E^n = M(X, CZ^{n,n}A) =$ the vector space of simplicial maps from X to the underlying simplicial set of $CZ^{n,n}A$, with the induced differential. Consider the dual chain complex $E_* = \mathrm{Hom}(E^*, \mathbb{Z}/2)$ and the usual free acyclic chain complex P_* over $\mathbb{Z}/2 = \{1, T\}$ with two 'cells' e_n and Te_n in each grading n and differential $de_n = (1 - T)e_{n-1}$, n even, $de_n = (1 + T)e_{n-1}$, n odd (see Steenrod and Epstein [9]). Then one has available the usual construction of an acyclic minimal carrier $\mathscr{C} : P_* \otimes E_* \to E_* \otimes E_*$ (which uses an Eilenberg–Zilber theorem for E_*) see, for example, Mosher and Tangora [8, p. 15], and consequently we obtain an equivariant chain map $\phi : P_* \otimes E_* \to E_* \otimes E_*$ with dual $\phi^* : P_* \otimes E^* \otimes E^* \to E^*$; the first tensor product is that of a chain complex and a cochain complex, see Steenrod [10]. Then $\mathrm{sq}^i : E^n \to E^{n+i}$ is defined by

$$\mathrm{sq}^i(x) = \phi^*(e_{n-i} \otimes x \otimes x + e_{n-i+1} \otimes x \otimes dx)$$

see, for example, Kristensen [5], and gives rise to the Steenrod squares in cohomology.

Note that the simplicial vector spaces $Z^{n,n}A$ are of finite type so the above construction yields maps $\mathrm{sq}^i : Z^{n,n}A \to Z^{m,m}A, m = n + i$. Then using the techniques of Kristensen [5] we can construct (by varying sq^i within its homotopy class) commutative diagrams for $m = n + i$

$$\begin{array}{ccccccccc}
\cdots & \twoheadrightarrow & Z^{n,n}A & \rightarrowtail & CZ^{n,n}A & \twoheadrightarrow & Z^{n+1,m+1}A & \rightarrowtail & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{sq}^i} & & \downarrow & & \downarrow{\scriptstyle \mathrm{sq}^i} & & \\
\cdots & \twoheadrightarrow & Z^{m,n}A & \rightarrowtail & CZ^{m,m}A & \twoheadrightarrow & Z^{m+1,n+1}A & \rightarrowtail & \cdots
\end{array}$$

Having obtain these cochain operaions along the 'diagonal' one passes them on to all of A^{**} via the canonically defined cochain operations called the 'weight-shifters' in [1] or [2]. Consequently, since $A^{**}X = M(X, A^{**})$ = the set of simplicial maps from X to the underlying simplicial set of A^{**}, we obtain the Steenrod squares in the mod 2 de Rham cohomology of any simplicial set X.

References

[1] H.E.A. Campbell, Mod 2 De Rham forms and the Steenrod algebra, Thesis, University of Toronto, June 1981.

[2] H.E.A. Campbell, Mod 2 De Rham forms and the Steenrod algebra, submitted for publication.

[3] H. Cartan, Théories Cohomologiques, Inventiones Math. 35 (1976) 261–271.

[4] E. Curtis, Simplicial homotopy theory, Adv. in Math. 6 (1971) 107-209.

[5] L. Kristensen, On secondary cohomology operations, Math. Scand. 12 (1963) 57-82.

[6] E.Y. Miller, De Rham cohomology with arbitrary coefficients, Topology 17 (1978) 193-203.

[7] J.P. May, Simplicial Objects in Algebraic Topology (Von Nostrand, New York, 1967).

[8] R. Mosher and M. Tangora, Cohomology Operations and Applications in Homotopy Theory (Harper and Row, New York, 1968).

[9] N. Steenrod and D.B.A. Epstein, Cohomology Operations, Annals of Math. Studies 50.

[10] N. Steenrod, Cohomology operations derived from the symmetric group, Comment. Math. Helv. 31 (1957) 195-218.